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On the Phelps–Koopmans theorem ☆

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Dedicated to the memory of David Cass: mentor, friend and an extraordinary economic theorist

Abstract

We examine whether the Phelps–Koopmans theorem is valid in models with nonconvex production technologies. We argue that a nonstationary path that converges to a capital stock above the smallest golden rule may indeed be efficient. This finding has the important implication that "capital overaccumulation" need not always imply inefficiency. Under mild regularity and smoothness assumptions, we provide an almost-complete characterization of situations in which every path with limit in excess of the smallest golden rule must be inefficient, so that a version of the Phelps–Koopmans theorem can be recovered. Finally, we establish that a nonconvergent path with limiting capital stocks above (and bounded away from) the smallest golden rule can be efficient, even if the model admits a unique golden rule. Thus the Phelps–Koopmans theorem in its general form fails to be valid, and we argue that this failure is robust across nonconvex models of growth.

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Keywords: Capital overaccumulation; Inefficiency; Phelps–Koopmans theorem; Nonconvex production set

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1. Introduction

The phenomenon of inefficiency of intertemporal consumption streams has been traditionally identified with the overaccumulation of capital. In fact, this message is strongly conveyed in two famous papers on efficiency by Malinvaud [4] and Cass [2].[1]

In the standard aggregative model of economic growth, the Phelps–Koopmans theorem provides one of the most well-known sufficient conditions for inefficiency.[2] This result was conjectured by Phelps [6], and its proof, based on a proof provided by Koopmans, appeared in Phelps [7]. It states that if the capital stock of a path is above, and bounded away from, the golden rule stock, from a certain time onward, then the path is inefficient.[3]

The purpose of this paper is to examine the validity of the Phelps–Koopmans theorem in aggregative models which allow for nonconvexity of the production set.[4] Of course, nonconvexity is no impediment to the existence of a golden rule provided that suitable end-point conditions hold (which we shall assume). Indeed, there may be several; we will refer to the smallest of them as the *minimal* golden rule. The Phelps–Koopmans theorem can then be restated in three progressively stronger formats:

I. Every stationary path with capital stock in excess of the minimal golden rule is inefficient.
II. A path is inefficient if it converges to a limit capital stock in excess of the minimal golden rule.
III. A path is inefficient if it is lies above (and bounded away from) the minimal golden rule from a certain time onwards.

Obviously, version III nests II, which in turn nests version I.

It is very easy to see that the weakest version I of the Phelps–Koopmans theorem must be true. But the analysis in Section 3.1 shows that version II of the theorem is generally false. We present there an example of an *efficient* path that converges to a limit stock that exceeds the minimal golden rule. This has the important implication that the phenomenon of "overaccumulation of capital" need not always imply inefficiency.

Since this finding might appear somewhat surprising, we try to convey an intuition for the result. Consider a setting with multiple golden rule stocks, and construct a path whose capital stock converges to some *nonminimal* (and therefore, by version I, inefficient) golden rule stock from above in such a way that at each period, the consumption level on the path in every period exceeds golden rule consumption.[5] If the path were inefficient, then there would be a path starting from the same initial stock, which dominates it in terms of consumption (in the efficiency ordering). This forces the capital stock of the dominating path to go below (and stay below) the

[1] In fact, one might make a case that this message can be inferred from the titles of the two papers.

[2] In awarding the Prize in Economic Sciences in Memory of Alfred Nobel for 2006 to Edmund Phelps, the Royal Swedish Academy of Sciences referred to this result as follows: "Phelps ... showed that all generations may, under certain conditions, gain from changes in the savings rate."

[3] The expression "overaccumulation of capital" in this literature refers therefore to accumulation of capital in excess of the golden rule capital stock in this precise sense. Thus, any stationary path with capital stock in excess of the golden rule capital stock, overaccumulates capital and is inefficient. The Phelps–Koopmans theorem generalizes this result to nonstationary paths.

[4] See Mitra and Ray [5] for a description of the setting, which does not assume smoothness of the production function, and does not place restrictions on the types of nonconcavities allowed.

[5] The consumption levels must, of course, converge to the golden rule consumption level over time.

inefficient golden rule stock after a finite number of periods. The nonconvexity in the production set now comes into play.

Suppose that the production function is such that the only golden rule stock below our inefficient golden rule stock is the minimal golden rule stock. Suppose, moreover, that the "curvature" of f at this minimal golden rule is larger than the corresponding curvature of f at the larger golden rule. That creates a lower surplus to the right of the minimal golden rule than at the larger rule. To maintain levels of consumption that dominate the original path, then, the capital stocks along the dominating path must shrink relatively rapidly, ultimately falling below the minimal golden rule, whereupon it becomes infeasible in a finite number of periods.[6] Thus, no dominating path can exist, and the constructed path must be efficient.

In view of the example it is natural to inquire whether there are general conditions on the production function that characterize when version II of the Phelps–Koopmans theorem can be shown to be valid. Certainly, we would like to allow for situations in which multiple golden rule stocks can exist,[7] and we are specially interested in providing a testable condition on the production function that guarantees version II without further qualifications.

Proposition 1 provides such a condition, which involves the comparative local behavior of the production function across multiple golden rules. Loosely speaking, the condition requires that the marginal product of capital fall more slowly at the minimal golden rule than at any of the other golden rules. It is therefore a condition which compares the local *curvatures* of the production function at various golden rules. Parts (i) and (ii) of the proposition show that under mild regularity conditions, the condition is also *necessary*: when it fails, so does version II of the Phelps–Koopmans theorem.

Proposition 2 extends Proposition 1 to a still more general specification, and provides a sufficient condition for version II of the Phelps–Koopmans theorem. This condition is automatically satisfied when the production function is concave, which is the focus of the traditional Phelps–Koopmans theory.[8] The sufficient condition is also implied by the condition of Proposition 1.

Finally, we examine version III of the theorem, which is the Phelps–Koopmans result in its strongest form. We show that this version of the theorem is generally false with or without the sufficient condition used to establish version II (Proposition 3 and Observation 1). Indeed, we prove that the version III is generally false even when there exists a unique golden rule. An interesting research question is to describe conditions under which version III is valid. We suspect that such conditions will involve strong restrictions on the production technology. Whether those conditions usefully expand the subset of convex technologies remains an open question.

2. Preliminaries

We begin by describing an aggregative model of growth. At every date, capital k_t produces output $f(k_t)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is the production function. We assume throughout that f satisfies the following restrictions:

[6] In Section 3.1, we elaborate further on this point.

[7] We know that in the case of an S-shaped production function, the theorem is valid (see Majumdar and Mitra [3]). However, in that setting, there is a unique golden rule stock, which occurs in the concave region of the production function, so that the traditional argument (used in models with concave production functions) applies.

[8] More precisely, the traditional Phelps–Koopmans theory assumes that the production function is *strictly* concave, so that there is a unique golden rule. But the condition nevertheless holds for production functions which are weakly concave.

[F.1] f is increasing and continuous, with $f(0) = 0$.
[F.2] There is $K \in (0, \infty)$ such that $f(x) > x$ for all $x \in (0, K)$ and $f(x) < x$ for all $x > K$.

We refer to K as the *maximum sustainable stock*. Observe that f is permitted to be nonconcave.

A *feasible path* from $\kappa \geqslant 0$ is a sequence of *capital stocks* $\{k_t\}$ with

$$k_0 = \kappa \quad \text{and} \quad 0 \leqslant k_{t+1} \leqslant f(k_t)$$

for all $t \geqslant 0$. Associated with the feasible path $\{k_t\}$ from κ is a *consumption sequence* $\{c_t\}$, defined by

$$c_t = f(k_{t-1}) - k_t \quad \text{for } t \geqslant 1.$$

It is obvious that for every feasible path $\{k_t\}$ from κ, both k_t and c_{t+1} are bounded above by $\max\{K, \kappa\}$. With no real loss of generality, we presume that $\kappa \in [0, K]$, so that for every feasible path $\{k_t\}$ from κ,

$$k_t \leqslant K \quad \text{for } t \geqslant 0 \quad \text{and} \quad c_t \leqslant K \quad \text{for } t \geqslant 1.$$

A feasible path $\{k'_t\}$ from κ *dominates* a feasible path $\{k_t\}$ from κ if

$$c'_t \geqslant c_t \quad \text{for all } t \geqslant 1,$$

with strict inequality for some t.

A feasible path $\{k_t\}$ from κ is *inefficient* if there is a feasible path $\{k'_t\}$ from κ which dominates it. It is *efficient* if it is not inefficient. A capital stock $k \in [0, K]$ will similarly be called inefficient if the corresponding stationary feasible path with $k_t = k$ for all t is inefficient; otherwise it is efficient.

Under [F.1] and [F.2] there is $k \in (0, K)$ such that

$$f(k) - k \geqslant f(x) - x \quad \text{for all } x \geqslant 0.$$

Then we call k a *golden rule stock*, or simply a *golden rule*. Certainly, there can be several golden rule stocks, all in $(0, K)$. Let G be the set of all golden rules. Obviously, G is nonempty and compact and so has a minimal element, which we denote by γ. *Golden rule consumption* is, of course, the same for all golden rules; it is given by $[f(k) - k]$ for $k \in G$, and is denoted by c.

It is easy to prove that the minimal golden rule is efficient. It is also easy to see that any capital stock that exceeds the minimal golden rule is inefficient. So version I of the Phelps–Koopmans theorem (see Introduction) must be true.

3. Phelps–Koopmans version II

3.1. An example

We begin with an example in which (i) there is an inefficient stock that exceeds the minimal golden rule, but (ii) there is an *efficient* path along which capital stocks converge to this inefficient stock. This example shows that it is possible to have higher capital stocks for all time periods compared to the capital stocks of an inefficient path, and still be efficient. Thus, version II of the Phelps–Koopmans theorem (see Introduction) breaks down, and the overaccumulation of capital does not translate into consumption inefficiency.

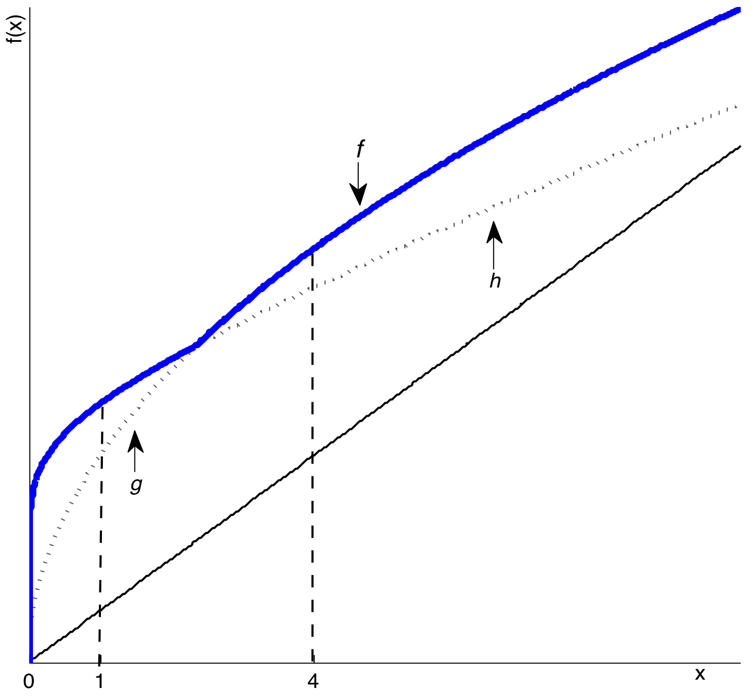

Fig. 1. The function f defined in Eq. (1).

It should be clear (and will become obvious in the analysis below) that the inefficient stock in any such example must itself be a golden rule.

We will define our technology to be the upper envelope of two techniques of production, each subject to diminishing returns in the input x. The first technique is described by

$$h(x) = \left[9x^{1/9} + x\right]/2 \quad \text{for all } x \geqslant 0$$

and the second by

$$g(x) = 4x^{1/2} \quad \text{for all } x \geqslant 0.$$

Define f as the pointwise maximum of these two techniques:

$$f(x) = \max\{h(x), g(x)\} \quad \text{for all } x \geqslant 0. \tag{1}$$

It is easy to see that f admits exactly two golden rules. Fig. 1, which summarizes f, provides visual "proof." The smaller (and minimal) rule γ is located at the unique solution to $h'(k) = 1$, given by $\gamma = 1$. The larger rule — call it ρ — is located at the unique solution to $g'(k) = 1$, which is at $\rho = 4$. Both golden rules generate a common value of golden rule consumption of 4. More generally, f satisfies [F.1] and [F.2].

We claim that, in this example:

There exists an efficient path with capital stocks that converge to a limit strictly in excess of the minimal golden rule.

In Proposition 1, we explicitly construct paths for a class of models that include this example and more, so we offer here only an informal argument that reveals how the construction works. Pick an initial stock above $\rho = 4$, and construct a path converging *down* to the golden rule ρ. It is easy to construct that path so that two properties are satisfied: (i) the path generates consumption strictly in excess of golden rule consumption at every date, and (ii) the path satisfies David Cass's celebrated criterion for efficiency *in a world where the production function is given by the technique g alone*. (While we invoke the Cass criterion in the formal arguments that follow, it is unnecessary to give a detailed account of it here.)

This latter requirement does not automatically guarantee efficiency in our model, of course, but it forces any dominating path — if one exists — to fall below the golden rule ρ after a finite number of periods. Indeed, were g to be the entire production function, that sequence of capital stocks would become infeasible, but in the case at hand g is not the entire production function; the technique h is also available. It is easy to see that the (presumed) dominating path must converge down to the golden rule at $\gamma = 1$: this is the only chance it has at dominating the original path at all consumption points. We now arrive at the heart of the example: the curvature of f at the minimal golden rule, as measured by the absolute value of the second derivative of h at $\gamma = 1$, is easily seen to be larger than the corresponding curvature of f at the larger golden rule ρ.[9]

The implication of the larger curvature at the lower golden rule γ is that the production function f generates a lower surplus (locally) to the right of the minimal golden rule γ than it does at its larger counterpart, ρ. To maintain levels of consumption that dominate the original path, then, the capital stocks along the dominating path must shrink relatively rapidly. The essence of the argument consists in showing that this relative rapidity eventually forces the dominating path to fall below the minimal golden rule, whereupon it becomes infeasible in a finite number of periods.

The fact is that when this ranking of curvature holds, an efficient path that converges to a limit that exceeds the minimal golden rule can *always* be constructed. Conversely, if this ranking of curvature is reversed, such an efficient path can *never* be constructed. This is what we turn to next.

3.2. A general result for version II

In this section, we substantially generalize the "multiple techniques" example of the previous section. Consider the following assumption, which imposes local smoothness on f at every golden rule, and some mild regularity conditions, while still allowing for several golden rules.

[F.3] f admits a finite set G of golden rules, it is C^2 in some interval around each of them, and $f''(k) < 0$ for every $k \in G$.

Under the additional restriction [F.3], we are interested in characterizing those functions f with the

Phelps–Koopmans property, version II. Let $\{k_t\}$ be any feasible path from κ with $\lim_{t\to\infty} k(t) = k > \gamma$. Then $\{k_t\}$ is inefficient.

9 The two curvatures are $4/9$ and $1/8$ respectively.

Proposition 1. *Suppose that* [F.1]–[F.3] *hold.*

(i) *If* $|f''(\gamma)| < |f''(k)|$ *for all other* $k \in G$*, then the Phelps–Koopmans property, version II, is satisfied.*
(ii) *If* $|f''(\gamma)| > |f''(k)|$ *for some* $k \in G$*, then the Phelps–Koopmans property, version II, must fail.*

Proof. Part (i) will follow as a direct corollary of Proposition 2 and the discussion in Section 3.4 connecting the smooth case to Condition [C], introduced in the same section. We establish part (ii) here.

Pick the smallest $k \in G$ such that $|f''(\gamma)| > |f''(k)|$. We will construct an *efficient* path $\{k_t\}$ from some $\kappa \geqslant 0$ with $\lim_{t\to\infty} k(t) = k$, thus violating the Phelps–Koopmans property, version II.

Define $\bar{M} \equiv |f''(k)|$ and $\bar{m} \equiv |f''(\gamma)|$. Choose m, M, and M' with $\bar{m} > m > M > M' > \bar{M}$. Pick $b > 0$ such that f is C^2 on $[k', k' + b]$ for every golden rule $k' \in G$, and

$$0 < -f''(x) \leqslant M' \quad \text{for all } x \in [k, k+b] \tag{2}$$

while

$$-f''(x) \geqslant m \quad \text{for all } x \in \left[k', k'+b\right] \text{ and all } k' \in G \text{ with } k' < k. \tag{3}$$

The existence of such a $b > 0$ follows directly from [F.3] and the definitions of k, m and M'. Finally, pick a positive integer N such that

$$\frac{M'}{M}\left(\frac{N+1}{N}\right)^2 \leqslant 1 \tag{4}$$

and

$$1/MN < b. \tag{5}$$

Define a sequence $\{k_t\}$ by

$$k_t = k + \left[1/M(t+N)\right] \quad \text{for all } t \geqslant 0. \tag{6}$$

We claim that $\{k_t\}$ is a feasible path from $\kappa = k + (1/MN)$. To see this, note that for $t \geqslant 0$, there is $z_t \in [k, k_t]$ such that

$$\begin{aligned} &f(k_t) - k_{t+1} \\ &\quad = f\left(k + \left[1/M(t+N)\right]\right) - k - \left[1/M(t+N+1)\right] \\ &\quad = f\left(k + \left[1/M(t+N)\right]\right) - f(k) + f(k) - k - \left[1/(t+N+1)\right] \\ &\quad = f'(k)\left[1/M(t+N)\right] + (1/2)f''(z_t)\left[1/M(t+N)\right]^2 + c - \left[1/M(t+N+1)\right] \\ &\quad = \left[1/M(t+N)\right] - \left[1/M(t+N+1)\right] - (1/2)\left[-f''(z_t)\right]\left[1/M(t+N)\right]^2 + c \\ &\quad \geqslant \left[1/M(t+N)(t+N+1)\right] - (1/2)M'\left[1/M(t+N)\right]^2 + c \\ &\quad \geqslant \left[1/M(t+N+1)^2\right] - \left(M'/2M\right)\left[1/M(t+N)^2\right] + c \end{aligned} \tag{7}$$

where we use the fact that $k_t \in [k, k+b]$ for all $t \geqslant 0$ (see (5) and (6)).

Note that $1/M(t+N)^2 = [1/M(t+N+1)^2][(t+N+1)^2/(t+N)^2] \leqslant [1/M(t+N+1)^2][(N+1)/N]^2$, so that

$$\begin{aligned}(M'/2M)\big[1/M(t+N)^2\big] &\leqslant (M'/2M)\big[1/M(t+N+1)^2\big]\big[(N+1)/N\big]^2 \\ &\leqslant (1/2)\big[1/M(t+N+1)^2\big] \end{aligned} \tag{8}$$

for all $t \geqslant 0$, where the second inequality uses (4). Combining (7) and (8), we see that

$$\begin{aligned} c_{t+1} = f(k_t) - k_{t+1} &\geqslant \big[1/M(t+N+1)^2\big] - (1/2)\big[1/M(t+N+1)^2\big] + c \\ &= \big[1/2M(t+N+1)^2\big] + c \end{aligned} \tag{9}$$

for $t \geqslant 0$. Thus, $\{k_t\}$ is a feasible path from κ, and $c_{t+1} = f(k_t) - k_{t+1} > c$ for all $t \geqslant 0$. Moreover, by construction, k_t decreases over time and converges to k as $t \to \infty$.

We claim that $\{k_t\}$ is efficient. Suppose, on the contrary, that there is a feasible path $\{k'_t\}$ from κ such that $c'_{t+1} \geqslant c_{t+1}$ for all $t \geqslant 0$, and $c'_{t+1} > c_{t+1}$ for some $t \geqslant 0$. It is easy to see that $k'_t \leqslant k_t$ for all $t \geqslant 0$.

We claim, first, that $k'_t < k$ for some $t \geqslant 0$. If not, then $k'_t \in [k, k_t] \subseteq [k, b]$ for all $t \geqslant 0$. Because $f''(x) < 0$ for all $x \in [k, k+b]$ (see (2)), we can follow the method of Cass [2] to obtain:

$$\sum_{t=1}^{\infty} \prod_{s=0}^{t-1} f'(k_s) < \infty. \tag{10}$$

On the other hand, for every $t \geqslant 1$,

$$\begin{aligned} f'(k_t) &= f'(k) + f''(\xi_t)(k_t - k) \quad \text{for some } \xi_t \in \big[k, k'_t\big] \\ &\geqslant 1 - M(k_t - k) \\ &= 1 - \big[1/(t+N)\big] \\ &\geqslant 1 - (1/t), \end{aligned} \tag{11}$$

where the first inequality follows from (2) and $M' < M$. It follows from Raabe's test (see, e.g., Bartle [1, p. 298]) that

$$\sum_{t=1}^{T} \prod_{s=0}^{t-1} f'(k_s) \to \infty \quad \text{as } T \to \infty,$$

contradicting (10). This proves our claim that $k'_t < k$ for some $t \geqslant 0$.

Because $c'_t \geqslant c_t > c$ for all $t \geqslant 1$, k'_t is strictly decreasing over time. It therefore converges to some $k' \geqslant 0$. The fact that $\{k'_t\}$ dominates $\{k_t\}$ implies that $f(k') - k' \geqslant c$. It follows that $k' \in G$. By the claim just proved, $k' < k$. Moreover, we can find a date S such that for $t \geqslant S$, we have $k'_t \in (k', k' + b)$. In what follows we focus on $t \geqslant S$. For such dates,

$$\begin{aligned} k'_{t+1} = f\big(k'_t\big) - c'_{t+1} &= f\big(k'_t\big) - f\big(k'\big) + f\big(k'\big) - c'_{t+1} \\ &= f'\big(k'\big)\big(k'_t - k'\big) + (1/2) f''(\zeta_t)\big(k'_t - k'\big)^2 + \big(c - c'_{t+1}\big) + k' \quad \text{for some } \zeta_t \in \big[k', k'_t\big] \\ &\leqslant \big(k'_t - k'\big) - (m/2)\big(k'_t - k'\big)^2 + \big(c - c'_{t+1}\big) + k', \end{aligned}$$

where the last inequality uses (3). Consequently, defining $\beta_t \equiv k'_t - k'$ for all t, we see that

$$\beta_{t+1} \leqslant \beta_t - (m/2)\beta_t^2 - \big(c'_{t+1} - c\big) \tag{12}$$

for all $t \geqslant S$. Now we know from (9) that $[1/2M(t+N+1)^2] \leqslant c_{t+1} - c \leqslant c'_{t+1} - c$, so using this in (12), we must conclude that for all $t \geqslant S$,

$$\beta_{t+1} \leqslant \beta_t - (m/2)\beta_t^2 - \left[1/2M(t+N+1)^2\right]. \tag{13}$$

Moreover, we know that $k_t' > k'$ for all $\geqslant 0$, so for all dates,

$$\beta_t > 0. \tag{14}$$

The following lemma completes the proof.

Lemma 1. *Consider the following difference inequality*:

$$x_{t+1} \leqslant x_t - px_t^2 - \left[q/(t+L)^2\right] \tag{15}$$

for $t \geqslant 0$, where p and q are strictly positive and L is a positive integer. This inequality admits a solution with $x_t > 0$ for all $t \geqslant 0$ only if $4pq \leqslant 1$.

For the proof, see Appendix A.

To complete the proof of the proposition, observe that (13) is a special instance of the difference inequality in (15), with $p = m/2$, $q = 1/2M$ and $L = N + 1$. Because $m > M$, we have $4pq > 1$. By Lemma 1, no solution to this difference inequality is possible with $\beta_t > 0$ for all t. □

3.3. The example revisited

It is easy to verify that the example in Section 3.1, with two techniques of production and two associated golden rules, satisfies [F.1]–[F.3]. It is also easy to see that in the example, the condition in part (ii) of Proposition 1 is satisfied. Therefore the Phelps–Koopmans property, version II, fails, which is the content of that example.

The example, as well as the proposition, suggests the following economic interpretation of the conditions in Proposition 1. Suppose that an economy has two or more "techniques" of production, each a concave differentiable function of variable input over and above some fixed level (the fixed cost). Define the *optimal scale* of a technique to be the value of the input at which net output, after subtracting the fixed and variable costs, is maximized. Define the marginal rate of return under an input scale k to be $f'(k)$. Say that technique a is *more scale sensitive* than technique b if the rate of return to a falls more rapidly (than that for b) for equal changes in the scale around the optimal scale. A technique which can be operated at different scales equally well has no scale sensitivity under this definition.

Our production function is obtained by constructing the outer envelope of these various techniques. It is easy to prove the following claim:

If techniques with low fixed costs are more scale sensitive compared to techniques with high fixed costs, then the Phelps–Koopmans property, version II, fails. Conversely, if the scale sensitivity comparison is reversed, the property holds.

3.4. An extension

One may be interested in a yet more general case in which [F.3] is not satisfied. For instance, there may be a continuum of golden rules, as in the case of production functions with an affine segment, or the production function may not be differentiable. We state a sufficient condition for the Phelps–Koopmans property (version II) to hold. Consider the following condition:

[C] For any golden rule $k > \gamma$, there is a golden rule $k' < k$ and $a > 0$ such that

$$f\big(k' + \epsilon\big) - f\big(k'\big) \geqslant f(k + \epsilon) - f(k) \quad \text{for all } \epsilon \in (-a, a). \tag{16}$$

Notice that if f is concave, and there is $k \in G$ with $k > \gamma$, then $[\gamma, k] \subset G$. Pick any $k' \in (\gamma, k)$, and pick $0 < a < \min\{k - k', k' - \gamma\}$. Then, for $\epsilon \in (-a, a)$, we have $k' + \epsilon \in (\gamma, k)$, so that $(k' + \epsilon) \in G$. Thus, $f(k' + \epsilon) - f(k') = f(k' + \epsilon) - (k' + \epsilon) + k' + \epsilon - f(k') = c - c + \epsilon = \epsilon$. On the other hand, $f(k + \epsilon) - f(k) = f(k + \epsilon) - (k + \epsilon) + k + \epsilon - f(k) \leqslant c - c + \epsilon = \epsilon$. Therefore [C] holds in this case.

On the other hand, if f satisfies [F.3], and the condition in part (i) of Proposition 1 is satisfied, Condition [C] *also* holds. Pick any golden rule $k > \gamma$. There exists $a > 0$ such that f is C^2 on $B(\gamma, a)$ and $B(k, a)$ and

$$\big[-f''(x)\big] < \big[-f''(z)\big] \quad \text{for all } x \in B(\gamma, a) \text{ and all } z \in B(k, a), \tag{17}$$

where $B(y, \epsilon)$ is the open ball of radius ϵ around y. Then, for $\epsilon \in (-a, a)$, we have

$$f(\gamma + \epsilon) - f(\gamma) = f'(\gamma)\epsilon + (1/2)f''(\xi)\epsilon^2 = \epsilon + (1/2)f''(\xi)\epsilon^2 \tag{18}$$

and

$$f(k + \epsilon) - f(k) = f'(k)\epsilon + (1/2)f''(\zeta)\epsilon^2 = \epsilon + (1/2)f''(\zeta)\epsilon^2, \tag{19}$$

where $\xi \in B(\gamma, \epsilon)$, and $\zeta \in B(k, \epsilon)$, as given by Taylor's theorem. Since $\xi \in B(\gamma, a)$ and $\zeta \in B(k, a)$ as well, we can use (18) and (19) to conclude that

$$f(\gamma + \epsilon) - f(\gamma) = \epsilon + (1/2)f''(\xi)\epsilon^2 > \epsilon + (1/2)f''(\zeta)\epsilon^2 = f(k + \epsilon) - f(k),$$

which establishes (16).

Thus one merit of Condition [C] is that it unifies the concave case as well as the locally smooth (but nonconcave) case studied in Section 3.2. Indeed, the condition is sufficient for version II of the Phelps–Koopmans property:

Proposition 2. *Suppose that* [F.1], [F.2] *and* [C] *hold. If* $\{k_t\}$ *is a feasible path from* κ *with* $\lim_{t\to\infty} k_t > \gamma$*, then* $\{k_t\}$ *is inefficient.*

Proof. Define $k \equiv \lim_{t\to\infty} k_t$. First suppose that k lies in G.

By [C], there is a golden rule $k' < k$ and $a > 0$ such that (16) holds. Denote $(k - k')$ by δ, $\min\{a, k'\}$ by b, and $(k_t - k)$ by ϵ_t for $t \geqslant 0$. Then, one can find $T \geqslant 0$ such that $\epsilon_t \in (-b, b)$ for all $t > T$. Define $k'_t = k_t$ for $0 \leqslant t \leqslant T$, and $k'_t = k' + \epsilon_t$ for $t > T$. Then, we have $k'_t \geqslant 0$ for all $t \geqslant 0$, and $c'_{t+1} = f(k'_t) - k'_{t+1} = f(k_t) - k_{t+1} = c_{t+1}$ for all $0 \leqslant t \leqslant T - 1$. Moreover, $c'_{t+1} = f(k'_t) - k'_{t+1} = f(k_t) - k_{t+1} + \delta = c_{t+1} + \delta > c_{t+1}$ for $t = T$. And for $t > T$, we have

$$\begin{aligned} c'_{t+1} &= f\big(k'_t\big) - k'_{t+1} = f(k_t - \delta) - (k_{t+1} - \delta) \\ &= f(k_t - \delta) - f(k_t) + f(k_t) - k_{t+1} + \delta \\ &= f(k_t - \delta) - f(k_t) + c_{t+1} + \delta. \end{aligned}$$

Thus, it is enough to show that $f(k_t - \delta) - f(k_t) + \delta \geqslant 0$ for all $t > T$.

Note that for $t > T$, we have $\epsilon_t \in (-b, b)$, so:

$$
\begin{aligned}
f(k_t-\delta)-f(k_t)+\delta &= f\bigl(k'+\epsilon_t\bigr)-f\bigl(k'\bigr)+f\bigl(k'\bigr)-f(k_t)+\bigl(k-k'\bigr)\\
&= f\bigl(k'+\epsilon_t\bigr)-f\bigl(k'\bigr)+c-f(k+\epsilon_t)+k\\
&= f\bigl(k'+\epsilon_t\bigr)-f\bigl(k'\bigr)+f(k)-f(k+\epsilon_t)\\
&\geqslant 0
\end{aligned}
$$

the last inequality following from (16).

This establishes the inefficiency of $\{k_t\}$ when $k \in G$.

If, on the other hand, $k \notin G$, then $f(k)-k<c$. Consequently, $c(t) \to f(k)-k<c$ as $t \to \infty$. Then one can easily dominate $\{k(t)\}$ by switching to the minimal golden rule γ sufficiently far in the future, and then staying at γ thereafter. □

4. Phelps–Koopmans version III

Version III of the Phelps–Koopmans theorem is false to a more robust degree.

4.1. A negative result

For some integer $n \geqslant 1$, an *n-cycle* $\mathbf{x}(n)$ is a vector of n stocks $\mathbf{x} \equiv (x_1, \ldots, x_n)$ such that

$$f(x_i) \geqslant x_{(i+1) \bmod (n)} \quad \text{for all } i=1,\ldots,n. \tag{20}$$

An n-cycle $\mathbf{x}$ is *weakly dominated* by an n-cycle $\mathbf{x}'$ if

$$f\bigl(x_i'\bigr)-x'_{(i+1) \bmod (n)} \geqslant f(x_i)-x_{(i+1) \bmod (n)} \quad \text{for all } i=1,\ldots,n.$$

Say that an n-cycle $\mathbf{x}'$ is *lower than* the n-cycle $\mathbf{x}$ if $x_i'<x_i$ for all $i=1,\ldots,n$.

Consider the following condition on f:

[F.4] There exists $n \geqslant 1$ and an n-cycle $\mathbf{x}$ such that (a) $x_i>\gamma$ for all i, and (b) $\mathbf{x}$ is not weakly dominated by any lower n-cycle.

In Section 4.2, we discuss [F.4] and provide a sufficient condition on the production function under which it will hold.

Proposition 3. *Whenever* [F.1], [F.2] *and* [F.4] *are satisfied, there exists an efficient path* $\{k_t\}$ *from some initial stock, with* $\inf_t k_t>\gamma$.

Proof. Fix an n-cycle $\mathbf{x}=(x_1,\ldots,x_n)$ as given by [F.4]. Define $\kappa \equiv x_1$ and let $\{k_t\}$ be a sequence from κ given by

$$k_t = x_{(t+1) \bmod (n)}$$

for all $t \geqslant 0$. Given (20), it is easy to see that this sequence forms a feasible path. By part (a) of [F.4], $\inf_t k_t>\gamma$. We are going to show that $\{k_t\}$ is efficient.

Let $\{c_t\}$ be the consumption sequence associated with $\{k_t\}$. Fix some $\theta \in (0,\kappa)$. We claim that there exists $\epsilon(\theta)>0$ such that for any feasible path $\{k_t'\}$ from any initial stock κ', with $\kappa' \in [0,\kappa-\theta]$ the inequalities

$$c_t' \geqslant c_t \tag{21}$$

for all $t = 1, \ldots, n$ must imply

$$k'_n \leqslant k'_0 - \epsilon(\theta). \tag{22}$$

Suppose that the claim is false. Then using the continuity of f, we must conclude that there exists a feasible path $\{k'_t\}$ from some $\kappa' \in [0, \kappa - \theta]$ such that (21) holds for all $t = 1, \ldots, n$, and such that $k'_n \geqslant k'_0$.[10] Note that for all $t = 0, \ldots, n-1$,

$$f(k'_t) - k'_{t+1} \geqslant f(k_t) - k_{t+1},$$

so that if $k'_t < k_t$, then $k'_{t+1} < k_{t+1}$. Because $k'_0 = \kappa' < \kappa = k_0$, we must conclude that $k'_t < k_t$ for every $t = 0, \ldots, n-1$. Moreover, because $\{k'_t\}$ is a feasible path and $k'_n \geqslant k'_0$, we see that the vector $\mathbf{x}' = (x'_1, \ldots, x'_n) \equiv (k'_0, \ldots, k'_{n-1})$ is an n-cycle, which is lower than $\mathbf{x}$.

At the same time, because (21) holds and because $k'_n \geqslant k'_0$, we see that

$$f(x'_i) - x'_{(i+1) \bmod (n)} \geqslant f(x_i) - x_{(i+1) \bmod (n)} \quad \text{for all } i = 1, \ldots, n,$$

but this contradicts [F.4]. Therefore the claim is true.

Now we return to the main proof. Suppose that $\{k_t\}$ is not efficient. Then there is a feasible path $\{k'_t\}$ from κ with associated consumption stream $\{c'_t\}$ such that $c'_t \geqslant c_t$ for all t, with strict inequality for some t. Without loss of generality, we may presume that strict domination occurs in the first n dates. Then, by [F.1], it is easy to see that

$$k'_t < k_t \tag{23}$$

for all $t \geqslant n$, and in particular, there is $\theta \in (0, \kappa)$ such that

$$k'_n \leqslant k_n - \theta = \kappa - \theta. \tag{24}$$

We claim that for all positive integers τ,

$$k'_{n(\tau+1)} \leqslant k'_{n\tau} - \epsilon(\theta), \tag{25}$$

where $\epsilon(\theta)$ is given by the claim above. To prove this claim, suppose (recursively) that $k_{n\tau} \leqslant \kappa - \theta$. Note that the sequence $\{k''_t\}$, defined by $k''_t = k'_{t+n\tau}$ for $t \geqslant 0$, is a feasible path from $k_{n\tau}$. We already know that $c'_t \geqslant c_t$ for all t, but because the sequence $\{c_t\}$ is cyclical with periodicity n, we also have that for every $t = 1, \ldots, n$,

$$c''_t = c'_{t+n\tau} \geqslant c_{t+n\tau} = c_t.$$

Therefore (21) holds for the path $\{k''_t\}$ from $k_{n\tau} \in [0, \kappa - \theta]$, and so by the claim, we must conclude that $k''_n \leqslant k''_0 - \epsilon(\theta)$. But this just means that

$$k'_{n(\tau+1)} \leqslant k'_{n\tau} - \epsilon(\theta),$$

which is (25). Note moreover that $k'_{n(\tau+1)} \leqslant k'_{n\tau} \leqslant \kappa - \theta$, so that the recursive argument can be continued. This proves the claim.

But now we have a contradiction, for no path satisfying (25) for all positive integers τ can be feasible. □

[10] The negation of (22) implies that there is a sequence $\epsilon^s \downarrow 0$ and a corresponding sequence (in s) of feasible paths $\{k^s_t\}$, each from some $k^s_0 \in [0, \kappa - \theta]$, such that for each s, (a) $c^s_t \geqslant c_t$ for $t = 1, \ldots, n$ and (b) $k^s_n \geqslant k^s_0 - \epsilon^s$. Using a diagonal argument, extract a pointwise convergent subsequence to establish the existence of the path $\{k'_t\}$.

4.2. Discussion

We now assess condition [F.4]. There are two important features of the discussion that follows. First, [F.4] is fully compatible with production functions that have a *unique* golden rule. Thus version III of the Phelps–Koopmans theorem can fail even in such situations. Second, [F.4] is not a "strong" condition: it could hold fairly naturally and robustly.

To make these points, we return to the "multiple techniques" example introduced in Section 3.1. Begin with $h(x)$ and $g(x)$, each of which satisfy [F.1] and [F.2]. In addition, assume that each function is strictly concave on $\mathbb{R}_+$ and C^2 on $\mathbb{R}_{++}$, with $h''(x) < 0$ and $g''(x) < 0$ for all $x > 0$. As in (1), define the production function f as the pointwise maximum of these two functions:

$$f(x) = \max\{h(x), g(x)\}$$

for all $x \geqslant 0$. We shall suppose that first h, then g, occupies the envelope; i.e., there exists u such that

$$f(x) = h(x) > g(x) \quad \text{for } x < u \quad \text{and} \quad f(x) = g(x) > h(x) \quad \text{for } x > u, \tag{26}$$

and we also suppose that

$$f(u) = h(u) = g(u) > u. \tag{27}$$

This second requirement guarantees that f satisfies [F.1] and [F.2].

Next, we suppose that the (unique) golden rules of the techniques h and g lie on either side of u:

$$\arg\max\{h(x) - x\} < u < \arg\max\{g(x) - x\}. \tag{28}$$

Finally, we impose the condition that there is just one golden rule for f:

$$\max\{h(x) - x\} > \max\{g(x) - x\}. \tag{29}$$

Fig. 2 illustrates the situation. The environment described in (26)–(29) seems to us to be quite natural.

We are now in a position to illustrate [F.4]. Let $U \equiv \arg\max\{g(x) - x\}$. Now consider the restriction:

$$g(U) - g(u) > u. \tag{30}$$

Observation 1. Under the production environment illustrated by (26)–(29), Condition [F.4] is implied by (30).

Proof. Given (30), pick v and V in (u, U) such that $v < V$ and

$$g(V) - g(v) > v. \tag{31}$$

It is easy to see that there exists $\epsilon > 0$ such that $g(x) - x > g(u) - u + \epsilon$ for all $x \in [v, V]$. Therefore there exists an integer n and a vector of stocks $\mathbf{x} = (x_1, \ldots, x_n)$ such that $x_1 = v$, $x_n = V$, and for every $i = 1, \ldots, n-1$, $x_i \leqslant x_{i+1} \leqslant x_i + \epsilon$, so that

$$g(x_i) - x_{i+1} = g(x_i) - x_i - (x_{i+1} - x_i) \geqslant g(u) - u. \tag{32}$$

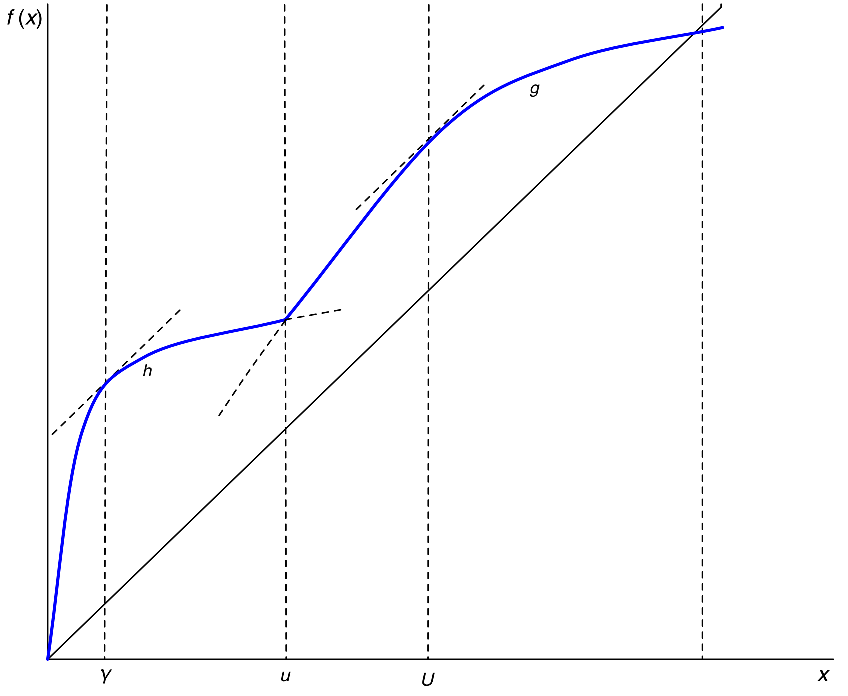

Fig. 2. The function $f(x) \equiv \max\{h(x), g(x)\}$.

Because $g(x_n) - x_1 = g(V) - v > g(v) > 0$, the vector $\mathbf{x} = (x_1, \ldots, x_n)$ forms an n-cycle.

Clearly, we have $x_i \geqslant v > u > \gamma$ for $i = 1, \ldots, n$, by (28). To complete the proof, we must show that the n-cycle $\mathbf{x}$ cannot be weakly dominated by any lower n-cycle. Suppose, on the contrary, that there is such an n-cycle $\mathbf{x}'$. If $x'_n \leqslant u$, then by (31),

$$f(x'_n) - x'_1 \leqslant f(x'_n) \leqslant f(u) = g(u) < g(v) < g(V) - v = g(x_n) - x_1 = f(x_n) - x_1,$$

a contradiction to weak dominance. Therefore $x'_n > u$. If $x'_i \leqslant u$ for some $i < n$, then by weak dominance,

$$f(x'_i) - x'_{i+1} \geqslant f(x_i) - x_i \geqslant g(u) - u = h(u) - u \geqslant f(x'_i) - u,$$

so that $x_{i+1} \leqslant u$ as well. This means we must have $x'_n \leqslant u$, contradicting the result just established. Therefore $x'_i > u$ for all $i < n$ as well.

Therefore such an n-cycle must have $x'_i > u$ for all i. Then using the strict concavity of f for all $x > u$, the fact that $f'(x_i) \geqslant 1$,[11] and $x_i > x'_i$ for $i = 1, \ldots, n$, weak dominance implies

$$x_{i+1} - x'_{i+1} \geqslant f(x_i) - f(x'_i) > f'(x_i)(x_i - x'_i) \geqslant (x_i - x'_i) \quad \text{for } i = 1, \ldots, n-1, \tag{33}$$

and

$$x_1 - x'_1 \geqslant f(x_n) - f(x'_n) > f'(x_n)(x_n - x'_n) \geqslant (x_n - x'_n). \tag{34}$$

[11] Recall that $x_i \leqslant V < U$ for all i, and that $f'(U) = 1$, so $f'(x_i) \geqslant 1$ for all i.

Adding (33) for $i = 1, \ldots, n-1$ and (34), we see that

$$\sum_{i=1}^{n}(x_i - x_i') > \sum_{i=1}^{n}(x_i - x_i')$$

a contradiction. Thus, $\mathbf{x} = (x_1, \ldots, x_n)$ cannot be weakly dominated by a lower n-cycle, establishing [F.4]. □

Remarks.

(i) The conditions (26)–(29) describe a natural environment, in which the upper envelope of two production techniques is chosen. The last condition (29) is deliberately imposed to strengthen the result: unlike version II, the failure of version III does not rely on the existence of multiple golden rules. Now, the restriction (30) does run in the opposite direction, requiring as it does the value of g to be "large" at its golden rule. But there is no contradiction across the two restrictions, and a robust set of environments satisfies both.

(ii) As a specific illustration, let

$$h(x) = 2x^{1/2} \quad \text{for all } x \geqslant 0$$

and

$$g(x) = (10/9)8^{1/10}x^{9/10} \quad \text{for all } x \geqslant 0.$$

It is straightforward to verify that all the conditions of Observation 1 (and therefore [F.4]) are satisfied.

(iii) The violation of the Phelps–Koopmans property, version III (for nonconvergent paths) is different from the violation of the Phelps–Koopmans property version II (for convergent paths) in one key respect.
In the latter, while the Phelps–Koopmans property can fail, its failure can be traced ultimately to (in fact characterized by) the behavior of the production function at its golden-rules. In the former, the behavior of the production function at its golden-rule(s) can cease to have any significance for the validity of the Phelps–Koopmans property.

(iv) Ray [8] extends the analysis of this section to show that there are paths that exceed and are bounded away from the unique golden rule that are *optimal* for some choice of one-period utility function and discount factor. Such paths are *a fortiori* efficient.[12]

4.3. A positive result for nonconvergent paths

Given the results of the preceding subsection, it appears difficult to make a general positive statement for nonconvergent paths. However, the following restatement of the Phelps–Koopmans theorem is valid even when the production set is nonconvex. This restatement is equivalent to the standard statement of the theorem when the production function is strictly concave.

In this section, we assume

[F.5] f is twice continuously differentiable on $\mathbb{R}_{++}$, with $f'(k) > 0$ for all $k > 0$.

[12] One might think that the converse must be generally true as well, but in fact it is not true of the convergent efficient paths that violate version II in Propositions 1 and 2 (see Ray [8]).

Say that a feasible path $\{k_t\}$ from κ is *interior* if $\inf_{t \geqslant 0} k_t > 0$.

Proposition 4. *Assume* [F.1], [F.2] *and* [F.5]. *Suppose that* $\{k_t\}$ *is an interior path from* $\kappa > 0$ *with*

$$\limsup_{t \to \infty} f'(k_t) < 1. \tag{35}$$

Then $\{k_t\}$ *is inefficient.*

Proof. Given (35), we have for $t \geqslant 1$,

$$\sum_{t=1}^{\infty} \prod_{s=0}^{t-1} f'(k_t) < \infty,$$

so by following the method of Cass [2, pp. 218–220], and noting that concavity of f is nowhere required, $\{k_t\}$ is inefficient. □

Remarks.

(i) This proof has been used in Majumdar and Mitra [3, p. 111, Theorem 3.2], under the assumption that f is "S-shaped."

(ii) Suppose f satisfies [F.1], [F.2], and [F.5], and moreover is strictly concave. Then there is a unique golden-rule γ. If $\{k_t\}$ is a feasible path from $\kappa > 0$ with

$$\liminf_{t \to \infty} k_t > \gamma,$$

then $\{k_t\}$ is an interior path from $\kappa > 0$ which satisfies (35), so that $\{k_t\}$ is inefficient. This is the standard version of the Phelps–Koopmans theorem.

Appendix A

Proof of Lemma 1. Suppose there is a strictly positive solution $\{x_t\}$ to (15). Define $y_t = t x_t$ for all $t \geqslant 0$. It can be checked that

$$\limsup_{t \to \infty} y_t \leqslant 1/p,$$

while

$$\liminf_{t \to \infty} y_t \geqslant q.$$

Let $R \equiv \liminf_{t \to \infty} y_t$. Then $R \in [q, 1/p]$.

Consider the quadratic equation

$$pz^2 - z + q = 0 \tag{36}$$

and suppose, contrary to the assertion of the lemma, that $4pq > 1$. Then the roots of (36) must be complex. Define for all $z \in \mathbb{R}$,

$$Q(z) = pz^2 - z + q.$$

Note that $Q(0) = q > 0$. Therefore it must be that $Q(z) > 0$ for all z. In particular, $Q(R) = pR^2 - R + q > 0$, and so we can find $\epsilon \in (0, R)$ such that

$$p(R - \epsilon)^2 - R + q > 0. \tag{37}$$

Because $R = \liminf_{t\to\infty} y_t$, there is $\bar{T}$ such that $y_t \geqslant R - \epsilon$ for all $t \geqslant \bar{T}$, or equivalently, $x_t \geqslant (R-\epsilon)/t$ for all $t \geqslant \bar{T}$. Using (15), and letting $r \equiv p(R-\epsilon)^2 + q$, we then have

$$r/(t+L)^2 \leqslant p\big[(R-\epsilon)^2/t^2\big] + \big[q/(t+L)^2\big] \leqslant px_t^2 + \big[q/(t+L)^2\big] \leqslant x_t - x_{t+1}, \qquad (38)$$

for all $t \geqslant \bar{T}$. For $T \geqslant \bar{T}$ and for any positive integer N, sum (38) from T to $T+N$ to get

$$\begin{aligned} x_T - x_{T+N+1} &\geqslant \sum_{t=T}^{T+N} \big[r/(t+L)^2\big] \geqslant \int_{T+L}^{T+N+L} \big(r/u^2\big)\,du \\ &= \big(r/(T+L)\big) - \big(r/(T+L+N)\big) \end{aligned}$$

so that

$$x_T \geqslant \big(r/(T+L)\big) - \big(r/(T+L+N)\big). \qquad (39)$$

Letting $N \to \infty$ in (39), we see that $x_T \geqslant r/(T+L)$, so that for all $T > \bar{T}$,

$$y_T \geqslant rT/(T+L),$$

and taking the lim inf on both sides of this inequality as $T \to \infty$, we see that $R \geqslant r$. Recalling the definition of r, this inequality implies that

$$p(R-\epsilon)^2 - R + q \leqslant 0$$

which contradicts (37). This contradiction shows that $4pq \leqslant 1$. □

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